ON THE FINAL MOTIONS OF CONSERVATIVE SYSTEMS

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We consider the final motions of Hamiltonian systems under the conditions of existence of the energy integral. Specifically, we consider natural conservative systems with k degrees of freedom, Liouville-type systems, homogeneous systems, and system which admit of a group of similarity transformations.

Imposing certain conditions on the potentials of the (not necessarily Newtonian) force fields, we obtain certain integrals and quasi-integrals of the equations of motion which contain secular terms.

A generalization of the Lagrange-Jacobi formula familiar in celestial mechanics is developed. This generalization turns out to be reducible to a nonlinear first-order differential equation by means of a certain invariant relation and the so-called configuration constant σ. The resulting equation is analyzed qualitatively and the most important cases of its integration are pointed out.

We also consider the conditions for stationary and quasi-stationary conservative systems as is done in the dynamics of stellar systems.

One of the problems of the qualitative theory of dynamic systems — the problem of final motions — arose in celestial mechanics in connection with studies of the Lagrange stability of the Solar System.

The problem was originally posed in terms of the changes in the relative distances between n gravitating point masses as the time t increases (decreases) without limit.

Studies on this problem were begun by Lagrange in his "Essai sur la problème des trois corps" [1].

The first theorems on the final motions of n bodies gravitating according to an arbitrary law were obtained by Jacobi [2] in connection with his studies of the stability of the Solar System.

Further studies of final motions were subsequently conducted largely in connection with the three-body problem. Chazy [3 and 4] attempted to analyze and classify these motions.

A general survey of final motions in the three-body problem will be found in [5]. Final motions in the n-body problem were studied by Khil'mi [6].

We shall consider the final motions of Hamiltonian systems under the condition of existence of the energy integral. The problem reduces to the investigation of the behavior of the representing point $N(q)$ in the k-dimensional space of configurations E^k as $t \to \pm\infty$.

As in stellar systems dynamics, the problem of the final motions of conservative systems can be reduced to the study of two functions. One of these functions, $\Omega = \Sigma p_j q_j$ is a bilinear form of canonical variables; the other function, J, is some function of the generalized coordinates q_j whose structure is similar to the expression for the moment of inertia of the system; this enables us to call it the "generalized moment of inertia". Specifically, for a free system of n gravitating point masses the value of J (in Cartesian coordinates) coincides exactly with the moment of inertia of the system.

1. Formulation of the problem and basic relations. Let some conservative system whose configuration is defined at each instant t by k generalized coordinates q_j $(j = 1, 2,\ldots, k)$ move in a force field with the potential $V(q)$. We assume that this potential is a homogeneous function of degree n of the generalized coordinates q_j, so that $V(\lambda q) = \lambda^n V(q)$.

We propose to consider the motion of the representing point $N(q)$ of the system in the k-dimensional space of configurations E^k whose metric is defined in Synge's terminology [7] by the kinematic linear element

$$dS^2 = 2T\,dt^2 = \sum_{i,\,j=1}^{k} a_{ij}(q)\,dq_i\,dq_j \qquad (a_{ij} = a_{ji}) \tag{1.1}$$

Here T is the kinetic energy of the system and $a_{ij}(q)$ are some functions of the coordinates q_j $(j = 1, 2,\ldots, k)$ which can be constants in a special case.

Let us write out Hamilton's equations,

$$\frac{dq_j}{dt} = \frac{\partial H}{\partial p_j}, \qquad \frac{dp_j}{dt} = -\frac{\partial H}{\partial q_j} \qquad (j = 1, 2, \ldots, k) \tag{1.2}$$

Multiplying these equations by p_j and q_j respectively, adding them together, and summing over the subscript j, we obtain

$$\frac{d}{dt}\sum_{i,\,j=1}^{k} p_j q_j = \sum_{i,\,j=1}^{k} \frac{\partial H}{\partial p_j} p_j - \sum_{i,\,j=1}^{k} \frac{\partial H}{\partial q_j} q_j \tag{1.3}$$

Following Poincare [8], we introduce the function Ω^* (Poincaré himself denotes this function simply by Ω without the asterisk) by way of the relation

$$\frac{d\Omega^*}{dt} = H(p, q) + \sum_{j=1}^{k} q_j \frac{dp_j}{dt} = H(p, q) - \sum_{j=1}^{k} \frac{\partial H}{\partial q_j} q_j \tag{1.4}$$

Combining Expressions (1.3) and (1.4) and noting that $\partial H / \partial p_j = \partial T^* / \partial p_j$, where $T^*(q, p)$ is the associated expression for the kinetic energy, we obtain

$$\frac{d}{dt}(\Omega - \Omega^*) = \sum_{j=1}^{k} \frac{\partial T^*}{\partial p_j} p_j - H(p, q) \qquad \left(\Omega = \sum_{j=1}^{k} p_j q_j\right) \tag{1.5}$$

Since, by virtue of (1.1), the kinetic energy $T(q, q^\cdot)$ of the system is a positive-definite quadratic form of the generalized velocities $q_j^\cdot$ it follows that the associated expression for the kinetic energy $T^*(q, p)$ is also a positive-definite quadratic form of the generalized impulses $p_j = \partial T / \partial q_j^\cdot$

$$T(q, q^\cdot) = \frac{1}{2}\sum_{i,\,j=1}^{k} a_{ij}(q)\,q_i^\cdot q_j^\cdot, \qquad T^*(q, p) = \frac{1}{2}\sum_{i,\,j=1}^{k} a^{ij}(q)\,p_i p_j \qquad (a^{ij} = a^{ji}) \tag{1.6}$$

Here $(a^{ij}) = (a_{ij})^{-1}$ is the inverse of the matrix (a_{ij}).

Making use of Euler's theorem on homogeneous functions and introducing the Lagrangian L, we transform Expression (1.5) into

$$d(\Sigma p_j q_j - \Omega^*) = L\,dt \qquad (L = T - V) \tag{1.7}$$

This result means that as the representing point $N(q)$ of the system moves along a straight path (in the space E^k metrized according to (1.1)), the elementary Hamiltonian operation $L\,dt$ represents the total differential of the difference $\Omega - \Omega^*$.

2. Natural systems. As we know, the Hamiltonian $H(p, q)$ for such systems can be represented as the sum of the kinetic and potential energies, i.e. as

$H(p, q) = T^*(q, p) + V(q)$.

Making use of the energy integral $H(p, q) = h$ and noting that $T^*(q,p)$ is a homogeneous quadratic form in the generalized impulses p_j, and $V(q)$ a homogeneous function of degree n of the generalized coordinates q_j, we apply Euler's theorem on homogeneous functions to transform (1.3) and (1.4) into

$$\frac{d\Omega}{dt} = -nh + (2+n)T^* - \sum_{j=1}^{k} \frac{\partial T^*}{\partial q_j} q_j, \quad \frac{d\Omega^*}{dt} = (1-n)h + nT^* - \sum_{j=1}^{k} \frac{\partial T^*}{\partial q_j} q_j \tag{2.1}$$

Let us assume that the associated expression for the kinetic energy $T^*(q, p)$ is a homogeneous function of degree $(-\nu)$ of the generalized coordinates q_i. By virtue of Euler's theorem on homogeneous functions we obtain

$$\frac{d\Omega}{dt} = -nh + (2+n+\nu)T^*, \qquad \frac{d\Omega^*}{dt} = (1-n)h + (n+\nu)T^* \tag{2.2}$$

Summing, we find that

$$\frac{d}{dt}(\Omega + \Omega^*) = (1-2n)h + 2(1+n+\nu)T^* \tag{2.3}$$

Theorem 2.1. Let a conservative system move in a force field whose potential energy $V(q)$ is a homogeneous function of degree n of the generalized coordinates q_j. Now let the associated expression of the kinetic energy $T^*(q, p)$ be a quadratic form in the generalized impulses p_j and a homogeneous function of degree $(-\nu)$ in the generalized coordinates q_j.

Then, if $(1+n+\nu) = 0$ it follows that there exists a relation of the form

$$\Omega^* = -(p_1q_1 + \dots + p_kq_k) + (1-2n)ht + \text{const} \tag{2.4}$$

which we call a "quasi-integral", since the function Ω^* itself is defined by a differential relation.

The existence of quasi-integral (2.4) which can be resolved into three components: the function Ω^*, the bilinear form Ω, and the secular term $(1-2n)ht$, follows directly from the integration of (2.3).

The condition $1+n+\nu = 0$ is fulfilled, for example, when the system consists of an arbitrary number of point masses gravitating in accordance with Newton's law. In this case we have $\nu = 0$, $n = -1$, so that the conditions of Theorem 2.1 are fulfilled. By virtue of (2.4), we infer that in this case

$$\Omega^* = -(p_1q_1 + \dots + p_kq_k) + 3ht + \text{const} \tag{2.5}$$

which coincides with the result obtained by Poincaré in [8].

Corollary 2.1. Let the conditions of Theorem 2.1 be fulfilled. Then for hyperbolic $(h > 0)$ and elliptic $(h < 0)$ motions the sum $(\Omega^* + \Omega)$ either increases or decreases monotonically with the time t, depending on $\operatorname{sign}(1-2n)h$.

In the case of parabolic motion $(h = 0)$ the secular term vanishes and the conservation law $\Omega^* + \Omega = \text{const}$ holds.

Theorem 2.2. Let the conditions of Theorem 2.1 concerning the homogeneity of the associated expression for the kinetic energy $T^*(q, p)$ and of the potential energy $V(q)$ be fulfilled.

Then, if the homogeneity exponents $(-\nu)$ and n satisfy the relation $(2+n+\nu) = 0$, there exists an integral which can be resolved into the bilinear form Ω and the secular term (nht),

$$\sum_{j=1}^{k} p_j q_j = -nht + \text{const} \qquad \left(p_j = \sum_{i}^{k} a_{ij}(q)\, q_i^{\cdot}\right) \tag{2.6}$$

This result follows directly from (2.2) and (1.5). Integral (2.6) can be written in another form with the aid of the energy integral. This yields

$$\sum_{i,j=1}^{k} a_{ij}(q)\, q_j q_i^{\cdot} + nt\left(\frac{1}{2}\sum_{i,j=1}^{k} a_{ij}(q)\, q_i^{\cdot} q_j^{\cdot} + V(q)\right) = \text{const} \tag{2.7}$$

which coincides with the familiar result [9].

Corollary 2.2. Let the conditions of Theorem 2.2 be fulfilled. Then for hyperbolic $(h > 0)$ and elliptic $(h < 0)$ motions the bilinear form $\Sigma p_j q_j$ either increases or decreases monotonically with the time t, depending on sign (nh). Hence, in final motion as $t \to \infty$ the function $\Omega(p(t),\ q(t))$ increases without limit in absolute value.

For parabolic motions $(h = 0)$ the secular term (nht) vanishes and the conservation law $\Omega(p,\ q) = \mathbf{const}$ holds.

Hence, there exists an integral of the form

$$\sum_{i,j=1}^{k} a_{ij}(q)\, q_j q_i = \text{const} \tag{2.8}$$

Theorem 2.3. Let the conditions of Theorem 2.1 concerning the homogeneity of the associated expression for the kinetic energy $T^*(q,\ p)$ and of the potential energy $V(q)$ be fulfilled.

Then, if the homogeneity exponents $(-\nu)$ and n satisfy the relation $n + \nu = 0$, there exists a quasi-integral of the form

$$\Omega^* = (1 - n)ht + \text{const} \qquad (h \neq 0) \tag{2.9}$$

The presence of the secular term $(1 - n)\,ht$ means that in the final motion as $t \to \infty$ the function increases without limit in absolute value.

The above result follows directly from (2.2). In particular, for a Newtonian gravitational force field we have $n = -1$, so that $\Omega^* = 2ht + \text{const}$

Theorem 2.4. Let the degrees of homogeneity of the potential energy $V(q)$ and of the associated expression for the kinetic energy $T^*(q,\ p)$ in the generalized coordinates q_j be n and $-\nu = 2$.

Then the bilinear form $\Sigma p_j q_j$ is a monotonic function of the time t in those domains G of the space E^k where the potential energy $V(q)$ is of strictly fixed sign ($V > 0$ or $V < 0$).

In order to see this we need merely use the energy integral and the condition $2 + \nu = 0$ to transform (2.2) into

$$d\Omega / dt = -nV(q) \tag{2.10}$$

Thus, if sign $(nV) > 0$, then $d\Omega / dt < 0$, and decreses monotonically; if sign $(nV) < 0$ we have $d\Omega / dt > 0$ and Ω increases monotonically.

In particular, for elliptic motions $(h < 0)$ the potential energy $V(q) = h - T(q,\ q^{\cdot}) < 0$, so that $\Omega(p,\ q)$ is a monotonically increasing or monotonically decreasing function of the time t.

For parabolic motions $(h = 0)$ the bilinear form Ω assumes stationary values at the stopping points $(T = 0)$. In any domain G not containing a stopping point we have

$V(q) = -T(q, q^{\cdot}) < 0$, so that Ω is also a monotonic function of the time t.

Now let us turn to (2.2) and establish the necessary conditions for the existence of periodic trajectories as the phase point $N^*(p, q)$ moves in the $2k$-dimensional phase space E^{2k}.

Introducing the average value of the kinetic energy $\langle T(t)\rangle$ over the finite time interval t, then integrating (2.2), we obtain

$$\Omega = (-nh + (2 + n + \nu)\ \langle T\rangle)\ t + \Omega_t$$
$$\Omega^* = ((1 - n)h + (n + \nu)\ \langle T\rangle)t + \Omega_0^* \tag{2.11}$$

Here $\Omega_0 = \Omega(0)$ and $\Omega^*{}_0 = \Omega^*(0)$ are constants, and

$$\langle T(t)\rangle = \frac{1}{t}\int_0^t T(q(t),\ p(t))dt \tag{2.12}$$

Let the phase point $N^*(p, q)$ for a given value of the constant energy h execute some periodic motion with the period τ in the phase space E^{2k} ; let this motion be such that $\Omega(\tau) = \Omega(0)$. Then, by (2.11),

$$\langle T(\tau)\rangle = \frac{nh}{2 + n + \nu} \qquad (h \neq 0) \tag{2.13}$$

Making use of the energy integral, we readily obtain from this a generalization of the familiar theorem on the virial: $(2 + \nu)\langle T\rangle = n\ \langle V\rangle$.

Let us turn to the problem of existence of periodic motions as determined by the values of the parameters h, ν and n.

1) Let $2 + \nu + n = 0$. Here periodic trajectories cannot exist for hyperbolic $(h > 0)$ and elliptic $(h < 0)$ motions. In the case of parabolic motion $(h = 0)$ periodic trajectories may occur; moreover, the bilinear form is conserved, i.e. $\Omega(p, q) = \Omega_0$.

2) Let $2 + \nu + n > 0$. The necessary (but not sufficient) condition for the existence of periodic trajectories in the case of hyperbolic $(h > 0)$ and elliptic $(h < 0)$ motions is the condition $\operatorname{sign}(nh) > 0$. Periodic trajectories cannot exist for parabolic motion $(h = 0)$.

3) Let $2 + \nu + n < 0$. The necessary (but not sufficient) condition for the existence of periodic trajectories in the case of hyperbolic $(h > 0)$ and elliptic $(h < 0)$ motions is $\operatorname{sign}(nh) < 0$. Periodic trajectories cannot exist for parabolic motions $(h = 0)$.

It is easy to verify these statements with the aid of (2.11) and (2.13).

In the general case of periodic motions $\Omega^*(t)$ is not a single-valued function of the variable t.

In fact, in traversing a periodic trajectory the value of Ω^* changes by the amount α during the period τ (by virtue of (2.13) and (2.11)).This quantity, which we call the "cyclical constant", is given by

$$\alpha = \frac{2 + \nu - n}{2 + \nu + n} h\tau \qquad (h \neq 0,\ \alpha = \Omega^*(\tau) - \Omega^*(0)) \tag{2.14}$$

If $2 + \nu - n = 0$, then the cyclical constant α vanishes and Ω^* becomes a single-valued function. As will be shown below, this condition holds in the case of Hamiltonian systems which admit of a one-parameter group of geometric similarity transformations of the form $q' = \lambda q$.

If the motion is parabolic $(h = 0)$ and periodic, then, as we have already shown, it is necessarily the case that $2 + n + \nu = 0$. Hence, by virtue of (2.11), the cyclical constant here is given by

$$\alpha = -2\tau\langle T(\tau)\rangle \qquad (h = 0) \tag{2.15}$$

For example, let us determine the cyclical constant α when the energy constant $h \neq 0$ and $h = 0$.

1. Let us consider Kepler's problem for a unit mass $(m = 1)$ moving in an elliptical orbit under action of an attracting center O with the Newtonian potential $V = -\mu / r$ (μ is the reduced mass). Placing the origin at the attracting center O and directing the x-axis along the line of apsides, we write the Hamiltonian., taking as our generalized coordinates the Cartesian coordinates of a point $(q_1 = x,\ q_2 = y)$,

$$H = (1 / 2\mu)(p_x^2 + p_y^2) - \mu / r \qquad (r = \sqrt{x^2 + y^2})$$

Here the associated expression for the kinetic energy T and the potential energy V are homogeneous functions of degree $\nu = 0$ and $n = -1$, respectively, in the generalized coordinates. Hence, by virtue of (1.4) we have

$$\frac{d\Omega^*}{dt} = h - \sum_{j=1}^{2} \frac{\partial H}{\partial q_j} q_j = h + V = h - \frac{\mu}{r} \qquad (2.16)$$

where the radius vector r (in accordance with the solution of the two-body problem) is given by

$$r = \frac{p}{1 + e\cos\vartheta} \qquad (p = a(1 - e^2)) \qquad (2.17)$$

Here a is the major semiaxis of the ellipse, e is the eccentricity, and ϑ is the true anomaly (the polar angle).

Integrating (2.16) over a single period τ and recalling that (by the area integral) $cdt = r^2 d\vartheta$, and that r is given by (2.17), we obtain

$$\alpha = \Omega^*(\tau) - \Omega^*(0) = \left(h - \frac{\mu}{p}\right)\tau - \frac{\mu e p}{c} \int_0^{2\pi} \frac{\cos\vartheta \, d\vartheta}{(1 + e\cos\vartheta)^2}$$

The integral in the right side of this expression can be computed with the aid of the Cauchy residue theorem (this entails the substitution $z = e^{i\vartheta}$ and conversion to complex variables). It turns out that

$$\int_0^{2\pi} \frac{\cos\vartheta \, d\vartheta}{(1 + e\cos\vartheta)^2} = -\frac{2\pi e}{(1 - e^2)^{3/2}}$$

Now, making use of the relations

$$c = \sqrt{\mu}\sqrt{a(1 - e^2)}, \qquad h = -\mu / 2a, \qquad \tau = 2\pi a^{3/2} / \sqrt{\mu}$$

known to us from the theory of elliptical orbits, we carry out certain simplifying operations to obtain

$$\alpha = h\tau\left(1 + \frac{2}{1 - e^2}\right) - h\tau \frac{2e^2}{1 - e^2} = 3h\tau$$

The same result can be obtained directly from (2.14) by setting $\nu = 0$, $n = -1$. This yields $\alpha = 3h\tau$, which is, in fact, correct.

2. For the case of parabolic motion $(h = 0)$ we can consider the periodic motion of a point of unit mass in a circular orbit under the action of an attracting center with the potential $V = -Ar^n$. From simple physical considerations we infer that $n = -2$, and direct computation of the cyclical constant α from (1.4) or (2.15) yields $\alpha = -\pi (8A)^{1/2}$.

3. Liouville-type systems. For these systems the kinetic energy T and the potential energy V are given by [11]

$$T(q, q') = \frac{U(q)}{2} \sum_{j=1}^{k} m_j(q_j) q_j'^2, \qquad T^*(q, p) = \frac{1}{2U(q)} \sum_{j=1}^{k} \frac{p_j^2}{m_j(q_j)} \qquad (3.1)$$

$$V(q)=\frac{1}{U(q)}\sum_{j=1}^{k}V_j(q_j),\quad U(q)=\sum_{j=1}^{k}U_j(q_j)$$

respectively.

Making use of the familiar method of [11], we convert from the canonical variables q_j and p_j to the new variables ξ_j and η_j $(j = 1, 2,\ldots, k)$ with the aid of point transformation (3.2) whose functional determinant D differs from zero,

$$\xi_j=\int\sqrt{m_j(q_j)}\,dq_j\qquad\left(D=\frac{\partial(\xi_1,\xi_2,\ldots,\xi_k)}{\partial(q_1,q_2,\ldots,q_k)}\neq 0\right)\tag{3.2}$$

The kinetic energy $T'^*(\xi,\eta)$ and potential energy $V'(\xi)$ in the new variables ξ_j and $\eta_j = \partial T' / \partial\xi_j^\cdot$ are given by

$$T'(\xi,\dot\xi)=\frac{U'(\xi)}{2}\sum_{j=1}^{k}\dot\xi_j^2,\quad T'^*(\xi,\eta)=\frac{1}{2U'(\xi)}\sum_{j=1}^{k}\eta_j^2\tag{3.3}$$

$$V'(\xi)=\frac{1}{U'(\xi)}\sum_{j=1}^{k}V_j(\varphi_j(\xi_j)),\quad U'(\xi)=\sum_{j=1}^{k}U_j(\varphi_j(\xi_j))$$

The above transformation is entirely canonical, so that the transformed Hamiltonian

$$H'(\xi,\eta)=T'^*(\xi,\eta)+V'(\xi)$$

follows from the initial function $H(p, q)$ by way of the substitution of variables

$$q_j=\varphi_j(\xi_j),\quad p_j=\sqrt{m_j'(\xi_j)}\,\eta_j\quad (m_j(\varphi_j(\xi_j))=m_j'(\xi_j))$$

so that $H(p, q) = H'(\xi,\eta) = h$.

Hamilton's equations (1.2) in the variables ξ_j and η_j become

$$\frac{d\xi_j}{dt}=\frac{\partial H'}{\partial\eta_j},\quad \frac{d\eta_j}{dt}=-\frac{\partial H'}{\partial\xi_j}\qquad (j=1,2,\ldots,k)\tag{3.4}$$

Let $U_j(q_j)$ and $V_j(q_j)$ be homogeneous functions of the variables q_j of degree λ and n, respectively, and $m_j(q_j)=a_jq_j^{\nu}\quad(\nu+2\neq 0,\ a_j=\mathrm{const})$

By virtue of transformation (3.2) we obtain

$$q_j=b_j\xi_j^{\theta}\qquad(\theta=2/(2+\nu),\ b_j=\mathrm{const})\tag{3.5}$$

so that $T'^*(\xi,\eta)$ and $V'(\xi)$ are homogeneous functions of the variables ξ_j $(j = 1, 2,\ldots, k)$ of degree $\lambda' = -\lambda\theta$ and $n' = (n-\lambda)\theta$, respectively.

This homogeneity of the functions $T'^*(\xi,\eta)$ and $V'(\xi)$ in the variables ξ_j is fulfilled for all values of ν except $\nu = -2$.

In this exceptional case, when $2+\nu = 0$, by virtue of (3.2) we obtain $\xi_j = C_j \ln q_j$, so that under conversion from the variables q_j to ξ_j (or vice versa) any power function ceases to be a power function after the above point transformation. For this reason we shall assume from now on that $2+\nu\neq 0$.

Turning to Eqs. (3.4) and proceeding as in Section 1, we recall the degrees of homogeneity of the functions $T'^*(\xi,\eta)$ and $V'(\xi)$ to obtain

$$\frac{d}{dt}\Omega'(\xi,\eta)=-n'h+(2+n'-\lambda')T'^*\qquad\left(\Omega'=\sum_{j=1}^{k}\xi_j\eta_j\right)\tag{3.6}$$

Let us introduce the function Ω'^*, setting

$$\frac{d\Omega'^*}{dt}=H'(\xi,\eta)-\sum_{j=1}^{k}\frac{\partial H'}{\partial\xi_j}\xi_j\qquad(H'(\xi,\eta)=h)$$

Making use of the energy integral and recalling the homogeneity of the function $H'(\xi, \eta)$ in the variables ξ_j, we apply Euler's theorem on homogeneous functions and arrive readily at the result

$$\frac{d\Omega'^*}{dt} = (1 - n')h + (n' - \lambda')T'^*(\xi, \eta) \tag{3.7}$$

Combinig (3.6) and (3.7) and introducing the Lagrangian

$$L'(\xi, \xi^\cdot) = T'(\xi, \xi^\cdot) - V'(\xi)$$

we obtain

$$\frac{d}{dt}(\Omega' - \Omega'^*) = L', \quad \frac{d}{dt}(\Omega' + \Omega'^*) = (1 - 2n')h + 2(1 + n' - \lambda')T'^* \tag{3.8}$$

These results are analogous to Expressions (1.7) and (2.3) for natural systems.

Theorem 3.1. Let $T'^*(\xi, \eta)$ and $V'(\xi)$ be homogeneous functions of the variables ξ_j of degree $\lambda' = -\lambda\theta$ and $n' = (n - \lambda)\theta$ for Liouville-type system (3.3). Then, if $2 + n' - \lambda' = 0$ (i.e. if $2 + n + \nu = 0$), then there exists an integral of the form

$$(\xi_1\eta_1 + \ldots + \xi_k\eta_k) + n'ht = \text{const} \qquad (n' = 2(n - \lambda)/(2 + \nu)) \tag{3.9}$$

which can be resolved into a bilinear form and a secular term.

This result follows directly from (3.6) and is analogous to integral (2.6) for natural systems.

Corollary 3.1. If the conditions of Theorem 3.1 are fulfilled, then periodic trajectories cannot exist for hyperbolic $(h > 0)$ and elliptic $(h < 0)$ motions; moreover, the bilinear form $\Sigma\xi_j\eta_j$ is then a monotonic function of the time t, becoming infinitely large in absolute value in the final motion as $t \to \infty$.

We assume here that $n \neq \lambda$, i.e. that $n' \neq 0$.

For parabolic motion $(h = 0)$ we have the conservation law $\Omega'(\xi, \eta) = \text{const}$, so that there exists an integral of the form

$$U'(\xi)(\xi_1\xi_1^\cdot + \xi_2\xi_2^\cdot + \ldots + \xi_k\xi_k^\cdot) = \text{const} \tag{3.10}$$

Theorem 3.2. Let the conditions of homogeneity of the functions $T'^*(\xi, \eta)$ and $V'(\xi)$ set forth in the condition of the Theorem 3.1 be fulfilled.

Then, provided $1 + n' - \lambda' = 0$, there esists a quasi-integral resolvable into three components, namely the function Ω'^*, the bilinear form $\Omega'(\xi, \eta)$ and the secular term $(1 - 2n')ht$;

$$\Omega'^* + (\xi_1\eta_1 + \xi_2\eta_2 + \cdots + \xi_k\eta_k) - (1 - 2n')ht = \text{const} \tag{3.11}$$

This result follows directly from (3.8) and is analogous to quasi-integral (2.4) for natural systems.

Now let us consider Liouville-type systems (3.1) when $m_j(q_j) = a_j = \text{const}$, and $U_j(q_j)$ and $V_j(q_j)$ are, as before, homogeneous functions of the variables q_j of degree λ and n, respectively.

Hence, setting $\nu = 0$ and making use of (3.5), we obtain $\theta = 1$, $n' = n - \lambda$, $\lambda' = -\lambda$, $q_j = b_j\xi_j$, so that basic relations (3.6), (3.7) and (3.8) become

$$\frac{d\Omega'}{dt} = -(n - \lambda)h + (2 + n)T'^*, \qquad \frac{d\Omega'^*}{dt} = (1 - n + \lambda)h + nT'^*$$

$$\frac{d}{dt}(\Omega' + \Omega'^*) = (1 - 2n + 2\lambda)h + 2(1 + n)T'^* \tag{3.12}$$

This means that for $n = -1$ there exists a quasi-integral which is resolvable into three terms, namely the function Ω'^*, the bilinear form $\Omega'(\xi, \eta)$, and the secular term $(3 + 2\lambda)ht$

$$\Omega'^* + (\xi_1\eta_1 + \xi_2\eta_2 + \cdots + \xi_k\eta_k) - (3 + 2\lambda)ht = \text{const} \tag{3.13}$$

which is a generalization of integral (2.5) obtained by Poincaré [8], and becomes the latter for $\lambda = 0$.

4. Periodic motions. Introducing the average value of the kinetic energy $\langle T'^* (t)\rangle$ over the finite time interval t, on integrating (3.6) and (3.7) we obtain

$$\begin{aligned} \Omega'(\xi, \eta) &= (-n'h + (2 + n' - \lambda') \langle T'^* (t) \rangle) t + \Omega' (0) \\ \Omega'^* (\xi, \eta) &= (1 - n') h + (n' - \lambda') \langle T'^* (t)\rangle) t + \Omega'^* (0) \end{aligned} \tag{4.1}$$

This implies that if periodic motions exist, then it is necessarily the case that

$$\operatorname{sign} (2 + n' - \lambda') = \operatorname{sign} (n'h)$$

Let the phase point of the system execute a periodic motion in the $2k$-dimensional phase space with a period τ such that $\Omega' (\tau) = \Omega' (0)$. Then, by (4.1), the average kinetic energy $\langle T'^*(\tau)\rangle$ over the period τ is given by

$$\langle T'^* (\tau)\rangle = \frac{n'h}{2 + n' - \lambda'} \tag{4.2}$$

Noting that $\lambda' = -\lambda\theta$, $n' = (n-\lambda)\theta$, $\theta = 2 / (2 + \nu)$, we can use the energy integral $\langle T'^* (\tau)\rangle + \langle V' (\tau)\rangle = h$ to obtain a virial relation of the form

$$(2 + \nu + \lambda) \langle T'^* (\tau)\rangle = (n - \lambda) \langle V' (\tau)\rangle \tag{4.3}$$

which is a generalization of our earlier relation for natural systems, and becomes the latter for $\lambda = 0$.

The function Ω'^* is non-singlevalued in the case of periodic motion and changes over the period τ by the cyclical constant α' given by

$$\alpha' = \frac{2 - \lambda' - n'}{2 - \lambda' + n'} h\tau \qquad (\alpha' = \Omega'^* (\tau) - \Omega'^* (0)) \tag{4.4}$$

All the results obtained in Sections 3 and 4 can be extended to systems more general (nonintegrable systems in the general case) than Liouville-type systems (3.1) and (3.3).

Precisely as regards $U(q)$ we need merely require that it be a homogeneous function of degree λ of the variables q_j, and not necessarily a superposition of homogeneous functions of the same degree λ of the form

$$U(q) = a_1 q_1^{\lambda} + a_2 q_2^{\lambda} + \ldots + a_k q_k^{\lambda} \qquad (a_j = \text{const})$$

as in the case of the Liouville-type systems considered above.

5. Homogeneous systems. This will be the term applied to systems whose potential energy $V(q)$ is a homogenous function of degree n in the variables q_j and whose kinetic energy $T(q, \dot q)$ has the same structure as Liouville-type systems (3.1), provided $U(q) \equiv 1$ and $m_j(q_j)$ are homogeneous functions of the same degree (which we denote by ν) of the variables q_j.

Hence, $T(q, \dot q)$ and $T^*(q, p)$ are of the form

$$T(q, \dot q) = \frac{1}{2} \sum_{j=1}^{k} m_j(q_j) \dot q_j^2, \quad T^*(q, p) = \frac{1}{2} \sum_{j=1}^{k} \frac{p_j^2}{m_j(q_j)} \tag{5.1}$$

Since the generalized impulse $p_j = m_j(q_j) \dot q_j$, it follows that,

$$p_j q_j = m_j(q_j) q_j \dot q_j = \frac{1}{2} \frac{d}{dt} (m_j(q_j) q_j^2) - \frac{1}{2} q_j^2 \frac{d}{dt} (m_j(q_j))$$

Summing over j and noting that $m_j(q_j)$ are homogeneous functions of degree ν in the variables q_j, with the aid of Euler's theorem on homogeneous functions we obtain

$$\frac{dJ}{dt} = (2+\nu)\sum_{j=1}^{k} p_j q_j \qquad \left(J = \sum_{j=1}^{k} m_j(q_j)\, q_j^2\right) \tag{5.2}$$

We call the quantity J the "generalized moment of inertia" of the system. In particular, for a free system consisting of gravitating point masses the quantity J assumes the usual form of the moment of inertia of the system.

Differentiating Eq. (5.2) and noting that $T^*(q, p)$ is a homogeneous function of degree $(-\nu)$ in the variables q_j, by virtue of (2.1) and (2.2) we obtain

$$\frac{d^2J}{dt^2} = (2+\nu)(-nh + (2+n+\nu)T^*) \tag{5.3}$$

which is a generalization of the Lagrange-Jacobi equation of celestial mechanics.

To see this, let us consider a free system consisting of N point masses m_i gravitating according to an arbitrary power law.

Here $m_i(q_j) = m_i = \text{const}$, $\nu = 0$, q_j are the Cartesian coordinates of the point m_i $(i = 1, 2, \ldots, N;\ j = 1, 2, \ldots, 3N)$, so that

$$\frac{d^2J}{dt^2} = -2nh + 2(2+n)T^* \qquad \left(J = \sum_{i=1}^{N} m_i r_i^2\right) \tag{5.4}$$

In the case of a Newtonian gravitational force field $(n = -1)$ this relation yields the familiar Lagrange-Jacobi equation [12 and 13].

It is sometimes expedient to rewrite Eq. (5.3) in a form such that the potential energy V appears instead of the kinetic energy T. Making use of the energy integral for this purpose, we obtain

$$\frac{d^2J}{dt^2} = (2+\nu)^2(h - mV) \qquad \left(m = \frac{2+\nu+n}{2+\nu},\ \nu \neq -2\right) \tag{5.5}$$

Theorem 5.1. Let conservative system (5.1) move in a Newtonian gravitational force field $(n = -1)$, and let the degree of homogeneity $(-\nu)$ of the associated expression of the kinetic energy $T^*(q, p)$ in the variables q_j be smaller than unity $(-\nu < 1)$. The bilinear form $\Sigma p_j q_j$ in the case of hyperbolic motions $(h > 0)$ is then a monotonically increasing function of the time t.

In fact, since $n = -1$, $h > 0$, and $1 + \nu > 0$, it follows by virtue of (5.3) that

$$\frac{d^2J}{dt^2} = (2+\nu)(h + (1+\nu)T) > 0$$

Hence, dJ/dt is an increasing function. By virtue of (5.2) this implies that $\Sigma p_j q_j$ is also an increasing function. The theorem has been proved.

Corollary 5.1. Theorem 5.1 is also valid for parabolic motions $(h = 0)$ provided we assume that the domain of the phase space under consideration does not contain quiescent points or stopping points.

6. Similar systems. In this category we place Hamiltonian systems invariant relative to a group of similarity transformations.

Let the homogeneous systems considered in Sect. 5 be similar and let them admit of a two-parameter group of similarity transformations of the form $q_j' = \lambda q_j$, $t' = \tau t$, where λ and τ are parameters. Since the kinematic energy $T(q, q^{\cdot})$ and potential energy $V(q)$ are transformable by means of Formulas

$$T'(q'q^{\cdot\prime}) = \lambda^{\nu+2}\tau^{-2}T(q, q^{\cdot}), \qquad V'(q') = \lambda^n V(q) \tag{6.1}$$

it follows that the parameters defining the above similarity transformation are related

by an expression [14] of the form
$$\lambda^{\nu+2-n}\tau^{-2}=1 \tag{6.2}$$
and that the new energy constant h' is related to the initial energy constant h by the expression $h' = \lambda^n h$.

Since $J(q)$ (5.2) is a homogeneous function of degree $(\nu+2)$ in the variables q_j, it can be transformed into
$$J'(q')=\lambda^{\nu+2}J(q) \qquad (\nu+2\neq 0) \tag{6.3}$$
so that, by (6.1), we have the invariant relation
$$V'(J')^{1-m}=VJ^{1-m}=\sigma \qquad \left(m-1=\frac{n}{2+\nu}\right) \tag{6.4}$$

Here σ is the so-called "configuration constant" [15].

Specifically, for a system of point masses gravitating according to Newton's law we have $\nu=0$, $n=-1$, $m=1/2$, so that invariant relation (6.4) becomes $J'(V')^2=JV^2=\sigma^2$, which coincides with the familiar result of [15].

Integrating (5.5) with the aid of invariant relation (6.4), we reduce the problem of finding $J(q)$ to a single quadrature,
$$\left(\frac{dJ}{dt^*}\right)^2=hJ-\sigma J^m+C \qquad (t^*=\sqrt{2}\,(2+\nu)\,t) \tag{6.5}$$

Here C is an integration constant and t^* is the reduced time.

Equation (6.5) can be interpreted as the kinetic energy integral for the motion of a point of unit mass over a given trajectory with the arc coordinate s under the action of some local force $f(s)$.

In fact, writing the equation of motion $s^{\cdot\cdot}=f(s)$, multiplying both sides by $2s^{\cdot}$, and integrating, we obtain
$$\left(\frac{ds}{dt}\right)^2=\Phi(s) \qquad \left(\Phi(s)=2\int f(s)\,ds+C\right) \tag{6.6}$$
which coincides with Eq. (6.5) if we take
$$s=J,\quad t=t^*,\quad \Phi(s)=hs-\sigma s^m+C$$

Separating variables and integrating, we obtain t^* as a function of J
$$t^*=\pm\int\frac{dJ}{\sqrt{\Phi(J)}}+\text{const} \qquad (\Phi(J)=hJ-\sigma J^m+C) \tag{6.7}$$

Let us cite the basic results of a qualitative analysis of Eq. (6.5) in accordance with the familiar method of Weierstrass [16], who carried out a general qualitative investigation of equations of the form (6.6).

We denote the increasing roots of the equation $\Phi(J)=0$ by $J_1<J_2<\ldots<J_p$.

1) Let the initial value of J, i.e. J_0, lie to the right of J_p (i.e. let $J_0>J_p$), and let the initial value of the derivative $(dJ/dt^*)_0$ be larger than zero. Since for $J\geqslant J_0$ the function $\Phi(J)$ is larger than zero and does not vanish, and since $dJ/dt^*>0$, it follows that $J(t^*)$ increases monotonically, varying from J_0 to $J=+\infty$, while the time $t^*=t_1^*$ is either finite or infinite, depending on whether the integral
$$t_1^*=\int_{J_0}^{\infty}\frac{dJ}{\sqrt{\Phi(J)}}$$
converges or diverges.

2) Let the initial value J_0 lie to the left of all the real roots of the equation $\Phi(J)=0$, i.e. let $J_0<J_1$, and let the derivative $(dJ/dt^*)_0$ be larger than zero as above. Since the function $\Phi(J)>0$ does not vanish anywhere in the interval $J_0\leqslant J<J_1$, it follows that $dJ/dt^*>0$ and that $J(t^*)$ increases monotonically, varying from the initial value

J_0 to some finite value J_1, which is the smallest root of the equation $\Phi(J) = 0$. The time t^* also increases monotonically, remaining finite if the root $J = J_1$ is simple, and becoming infinitely large if the root $J = J_1$ is multiple.

Thus, in the case of a simple root $J = J_1$ the quantity J varies monotonically from the initial value J_0 to reach its maximum value $J = J_1$ in the course of the finite time interval t_1^*, after which the motion is reversed, since $\Phi(J) = (J_1 - J)\,\Phi_1(J)$, so that for $J = J_1$ we obtain $(d\Phi(J)/dJ)_{J_1} = -\Phi_1(J_1) < 0 \qquad (\Phi_1(J) > 0)$

With further increases in the time $t^* > t_1^*$ the function J decreases monotonically, and in the final motion as $t^* \to +\infty$ we obtain $J \to -\infty$.

If the root $J = J_1$ is multiple, it follows that $t_1^* = \infty$ and J varies monotonically in its final motion, approaching the value J_1 as its upper bound (asymptotic motion).

3) Let the initial value $J = J_0$ lie between two simple roots of the equation $\Phi(J) = 0$, which (without limiting generality) we denote by J_1 and J_2 so that $J_1 < J_0 < J_2$, and

$$\Phi(J) = (J_2 - J)(J - J_1)\,\Phi_1(J) \qquad (\Phi_1(J) > 0)$$

Since $J_1 < J < J_2$, since the function $\Phi(J)$ is larger than zero, and since dJ/dt^* is of fixed sign, it follows that J increases monotonically from J_0 to J_2 if $(dJ/dt^*)_0 > 0$, or decreases monotonically from J_0 to J_1 if $(dJ/dt^*)_0 < 0$. The motion reverses at the stopping points $J = J_1$ and $J = J_2$, since the derivative $d\Phi/dJ$ changes sign at these points. Hence, the point J executes a periodic motion, assuming its largest and smallest values at the points $J = J_2$ and $J = J_1$, respectively.

4) Let $J_1 < J_0 < J_2$, and let the roots J_1 and J_2 be multiple. Then, depending on the sign of $(dJ/dt^*)_0$, the function $J(t^*)$ in the final motion as $t \to \infty$ asymptotically approaches one of the roots, namely $J = J_2$ if $(dJ/dt^*) > 0$ and $J = J_1$ if $(dJ/dt^*)_0 < 0$.

Let us cite some individual cases of integration of (6.5) depending on the values of the parameter m (5.5). Omitting the intervening computations, we shall merely set down the final results.

1) Let $m = 0$; then by virtue of (5.5), (6.2) and (6.4) we obtain

$$J = {}^1/_4 h t^{*2} + C_1 t^* + C_2 \qquad (2 + \nu + n = 0,\ \lambda^{2n}\tau^2 = 1)$$

Here C_1 and C_2 are integration constants.

2) Let $m = 1$. This yields $n = 0$, so that there is no external force field. If, moreover, $\nu = 0$, then $\lambda^2\tau^{-2} = 1$, and

$$J = {}^1/_2 C_0 t^{*2} + C_1 t^* + C_2 \qquad (C_0 = {}^1/_2 (h - \sigma),\ C_1,\ C_2 = \mathrm{const})$$

3) Let $m = 2$. This case corresponds to the one-parameter group of geometric similarity transformations,

$$q_j' = \lambda q_j, \quad \tau = \pm 1 \qquad (2 + \nu - n = 0,\ \lambda \neq 0)$$

Introducing the value $\Delta = h^2 + 4\sigma C$ and integrating (6.5), we obtain

$$J = A \sin \sqrt{\sigma}\,(t^* - t_0^*) + h/2\sigma \qquad (\sigma > 0,\ \Delta > 0,\ A = (1/2\sigma)\sqrt{\Delta})$$
$$J = A \operatorname{sh} \sqrt{-\sigma}\,(t^* - t_0^*) + h/2\sigma \qquad (\sigma < 0,\ \Delta < 0,\ A = -(1/2\sigma)\sqrt{|\Delta|})$$
$$J = A \operatorname{ch} \sqrt{-\sigma}\,(t^* - t_0^*) + h/2\sigma \qquad (\sigma < 0,\ \Delta > 0,\ A = -(1/2\sigma)\sqrt{\Delta})$$

Hence, for $\sigma > 0$ the function J remains bounded all the time, since it varies according to a harmonic oscillation law with the period $\tau^* = 2\pi/\sqrt{\sigma}$; for $\sigma < 0$ the function J increases without limit in the final motion as $t \to \infty$.

For example, let us consider a conservative system with k-degrees of freedom whose kinetic energy T and potential energy V are of the form

$$T=\frac{1}{2}\sum_{j=1}^{k} a_j q_j^{\nu} q_j'^2, \qquad V=\sum_{j=1}^{k} c_j q_j^{n}$$

Let the condition of the geometric similarity transformation be fulfilled for the system, so that $m = 2,\ 2 + \nu - n = 0$ and $\tau = 1$. Invariant relation (6.4) then becomes

$$\frac{V}{J}=\frac{c_1 q_1^{n}+c_2 q_2^{n}+\ldots+c_k q_k^{n}}{a_1 q_1^{\nu+2}+a_2 q_2^{\nu+2}+\ldots+a_k q_k^{\nu+2}}=\sigma \qquad (\nu+2=n)$$

or

$$(c_1-a_1\sigma)\,q_1^{n}+(c_2-a_2\sigma)\,q_2^{n}+\ldots+(c_k-a_k\sigma)\,q_k^{n}=0$$

Since the q_j are independent, we find from this that

$$\frac{c_1}{a_1}=\frac{c_2}{a_2}=\ldots=\frac{c_k}{a_k}=\sigma$$

The behavior of the function J as determined by the values of the parameters h, ν and σ can be studied with the aid of the above solutions.

4) Let $m = {}^1/_2$, so that by (6.4) and (6.2) we have

$$2n+\nu+2=0, \qquad \lambda^{-3n}=\tau^{2}$$

This case is of interest, since in the particular case of a free system of point masses gravitating in accordance with Newton's law we have $\nu = 0,\ n = -1$, so that the value of the parameter m is ${}^1/_2$. The condition of mechanical similarity for $n = -1$ becomes $\lambda^3 = \tau^2$, which, as we know, expresses Kepler's third law in the case of central motion.

Here we must distinguish among the cases $h < 0,\ h = 0$ and $h > 0$.

a) Let the motion be elliptic $(h < 0)$, and let the configuration constant σ be smaller than zero.

Making the substitution $\sigma / 2h - \sqrt{J} = k \cos E$, introducing the notation $\sigma_1 = -\sigma > 0,\ h_1 = -h > 0$, and integrating (6.5), we obtain Kepler's equation for elliptical orbits,

$$E-e\sin E=M \qquad \left(e=\frac{2hk}{\sigma},\ k^2=\frac{\sigma^2-4hC}{4h^2}>0,\ M=\frac{h_1^{3/2}}{\sigma_1}(t^*-t_0^*)\right) \quad (6.8)$$

Here E is the eccentric anomaly, M is the mean anomaly, e is the eccentricity, and the period $\tau_*^* = 2\pi\sigma_1 / h_1^{3/2}$ depends on the constant energy and the configuration constant σ.

In the case of parabolic motion $(h = 0)$ we have

$$u(u^2-3C)={}^3/_4\,\sigma^2(t^*-t_0^*) \qquad (u^2=C-\sigma\sqrt{J}) \qquad (6.9)$$

b) Let the motion be hyperbolic $(h > 0)$, and let the configuration constant σ be larger than zero.

Making the substitution $\sigma / 2h - \sqrt{J} = k \operatorname{ch} s$ and integrating (6.5), we obtain Kepler's equation for hyperbolic orbits,

$$e\operatorname{sh} s-s=M \qquad \left(e=\frac{2hk}{\sigma},\ k^2=\frac{\sigma^2-4hC}{4h^2}>0,\ M=\frac{h^{3/2}}{\sigma}(t^*-t_0^*)\right) \quad (6.10)$$

Let us apply the above relations to the two-body problem. Converting, as usual, to the problem of the central motion of a point with the reduced mass μ in a Newtonian field with the potential $V(r) = -A / r$, we write out the Lagrangian

$$L=\frac{\mu}{2}(x'^2+y'^2)+\frac{A}{r} \qquad \left(\mu=\frac{m_1 m_2}{m_1+m_2}\right)$$

In the case under consideration we have $\nu = 0, n = -1, m = {}^1/_2, J = \mu r^2$, and the configuration constant σ, by virtue of the invariance of (6.4), is equal to $\sigma = V\sqrt{J} = -A\sqrt{\mu}$; so that in the case of an attracting center $A > 0$ and $\sigma < 0$, while in the case of a repelling center $A < 0$ and $\sigma > 0$.

Turning to (6.5) and noting that $\nu = 0$, $t^* = 2\sqrt{2}t$ and $dJ / dt = 2\mu r\, dr / dt$, after certain simplifications we obtain

$$dr / dt = (2 / \mu\,(h + A / r - \mu c^2 / 2r^2))^{1/2} \qquad (C = -{}^1/_2\mu^2 c^2)$$

which coincides with the solution of [10].

In perticular, in the case $h < 0$ and $\sigma < 0$ we have

$$\sqrt{J} = \sigma / 2h - k\cos E = \sigma / 2h\,(1 - e\cos E)$$

Substituting in the values $J = \mu r^2$, $\sigma = -A\sqrt{\mu}$ and making use of the expression for the major semiaxis of the ellipse $a = -A / 2h$, we obtain Formula $r = a\,(1 - e\cos E)$ faniliar to us from the theory of elliptical orbits.

5) Let $m = 3$. Then $4 + 2\nu - n = 0$, $\lambda^{-n} = \tau^4$, and on integrating (6.5) we obtain:

for elliptic motion $(h < 0)$ for $\sigma < 0$ and $C < 0$,

$$J = \wp({}^1/_2\sqrt{-\sigma}\,t^* + \alpha) \qquad (g_2 = 4h/\sigma,\ g_3 = 4C/\sigma)$$

for hyperbolic motion $(h > 0)$ for $\sigma > 0$ and $C < 0$,

$$J = -\wp({}^1/_2\sqrt{\sigma}\,t^* + \alpha) \qquad (g_2 = 4h/\sigma,\ g_3 = -4C/\sigma)$$

Here $\wp(t^*;\ g_2, g_3)$ is the Weierstrass elliptic function, g_2 and g_3 are its invariants, and α is an integration constant.

6) Let $m = {}^1/_3$; then $4 + 2\nu + 3n = 0$, $\lambda^{-3n} = \tau^4$, and on integrating (6.5) we obtain:

for elliptic motion $(h_1 = -h > 0)$ for $\sigma < 0$ and $C < 0$,

$$t^* - t_0^* = \frac{2}{h_1}(h_1 z^3 + \sigma z + C)^{1/2} - \frac{2\sigma}{h_1^{3/2}}u(z) \qquad \left(J = -z^3,\ g_2 = \frac{4\sigma}{h},\ g_3 = \frac{4C}{h}\right)$$

for hyperbolic motion $(h > 0)$ for $\sigma > 0$ and $C < 0$,

$$t^* - t_0^* = \frac{2}{h}(hz^3 - \sigma z + C)^{1/2} + \frac{2\sigma}{h^{3/2}}u(z) \qquad \left(J = z^3,\ g_2 = \frac{4\sigma}{h},\ g_3 = -\frac{4C}{h}\right)$$

Here $u(z)$ is an elliptic integral whose inverse is the Weierstrass function $z = \wp(u)$.

In those cases where the constant $C > 0$, we can use the homogeneity formula for Weierstrass functions [17],

$$\wp(u;\ g_2, g_3) = \mu^2\wp(\mu u;\ g_2/\mu^4,\ g_3/\mu^6)$$

Hence, setting $\mu = i$ $(i = \sqrt{-1})$, we obtain

$$\wp(u;\ g_2, g_3) = -\wp(iu;\ g_2, -g_3)$$

We note that for the parameter values

$$m = 4\ (6 + 3\nu - n = 0,\ \lambda^{-n} = \tau^3), \qquad m = {}^1/_4\ (6 + 3\nu + 4n = 0,\ \lambda^{-3n} = \tau^6)$$

we can also integrate (6.5) in terms of elliptic functions.

7. Stationary points. The stationary values of the function J are, by (6.5), the roots of the equation

$$hJ - \sigma J^m + C = 0 \tag{7.1}$$

Denoting the corresponding values by J_i and making use of (6.5) and (6.4), we obtain

$$\left(\frac{d^2J}{dt^{*2}}\right)_{J=J_i} = \frac{h\,(1-m)\,J_i - mC}{2J_i} \tag{7.2}$$

This implies that at the stationary points, where $J_i \neq 0$, the function J assumes its minimum or maximum values depending on which of the following conditions is fulfilled:

$$h(1-m)J_i - mC > 0 \quad \text{or} \quad h(1-m)J_i - mC < 0.$$

If $h(1-m)J_i - mC = 0$ we have the case of a degenerate stationary point.

8. Stationary systems. Following the terminology of stellar systems dynamics [18 and 19], we call a system "statically stationary" if its generalized moment of inertia $J(q)$ remains constant, and "quasistationary" if $J(q)$ varies at a constant rate $dJ/dt = a$. In the latter case J is a linear function of t, i.e. $J = at + b$.

Theorem 8.1. Let the associated expression for the kinetic energy $T^*(q, p)$ for systems (5.1) be a homogeneous function of degree $(-\nu)$ in the variables q_j, let $2 + \nu \neq 0$, and let the potential energy $V(q)$ be a homogeneous function of degree n in the variables q_j.

Thus, if a system is statically stationary or quasistationary, then the kinetic energy T and the potential energy V are related by a virial expression of the form

$$(2+\nu)T = nV \tag{8.1}$$

This result follows directly from (5.3) if we make use of the energy integral.

In particular, for $\nu = 0$ we obtain the familiar theorem on the virial [10].

Corollary 8.1. Let the conditions of Theorem 8.1 be fulfilled. A system is then stationary or quasistationary if and only if the integral

$$p_1q_1 + p_2q_2 + \dots + p_kq_k = A \qquad (A = \text{const})$$

exists.

In fact, as we infer from (5.2), if $A = 0$, then $J = \text{const}$ and the system is stationary; if $A \neq 0$, then $J = at + b \; (a = (2+\nu)A)$ and the system is quasistationary.

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ONE-DIMENSIONAL UNSTEADY MOTIONS OF GAS DISPLACED BY A PISTON

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We consider the flow of a gas displaced by a piston which at some instant begins to expand according to a power law with an exponent smaller than that corresponding to an intense explosion. We assume that the gas has received a finite energy prior to the beginning of motion of the piston. The energy of the gas in this case remains finite over an infinite time interval, so that all of the required functions are obtainable by linearization relative to the values occurring in the problem of an intense explosion. The solution is constructed by investigating the inverse problem in which a shock wave moves through a quiescent gas of constant density and at a pressure negligibly small as compared with the pressure behind it is specified. The piston expansion law is obtained by solving the resulting Cauchy problem. Special attention is given to the case of a cylindrical piston of constant radius, when the required solution contains logarithmic terms.

The problem of motion of gas due to the expansion of a piston at a constant rate was solved by Sedov [1] and Taylor [2]. The more general case in which the velocity of the piston depends on time according to a power law was later investigated by Krasheninnikova [3] and by Kochina and Mel'nikova [4]. In these studies the functions describing the perturbed flow fields depend on the self-similar variable only and are found by integrating a system of nonlinear ordinary differential equations. As may be seen from qualitative investigation [3 and 4], the problem does not always have a solution if the piston motion is defined as $R = ct^n$ (where R is the coordinate and t is the time). In order for a solution to exist, the exponent n must satisfy the condition $n > 2 / (\nu + 2)$, where the parameter $\nu = 1, 2, 3$ for flows with plane, axial, and central symmetry, respectively.